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XVI.

THE POTENTIAL OF A SHELL BOUNDED BY CONFOCAL ELLIPSOIDAL SURFACES.

By Frank Nelson Cole.

Presented May 29th, 1883.

Let there be three confocal ellipsoids, of which the outer two have their bounding surfaces fixed, while that of the inner ellipsoid approaches that of the middle one, the three surfaces always remaining confocal. Let X_{13} and X_{23} be the respective attractions, resolved parallel to the axis of x, of the inner and middle ellipsoids at any point in the surface of the outer ellipsoid; and let X_{31} and X_{32} be the attractions of the outer ellipsoid at the corresponding points of the inner and middle ellipsoids. To calculate the limiting value of $X_{23} - X_{13}$ as the inner ellipsoid approaches the middle one:—

By Ivory's theorem,

$$X_{23} = \frac{b_2 c_2}{b_3 c_3} X_{32}.$$

Also,

$$X_{32} = 2 \reflectbox{∂} \rho a_3 b_3 c_3 Q_{a_3} x_2,$$

where

$$Q_{a_3} = \int_0^\infty \frac{dt}{(a_3^2 + t) \sqrt{(a_3^2 + t)(b_3^2 + t)(c_3^2 + t)}},$$

and a, b, c, and x are the semi-axes of an ellipsoid and the x co-ordinate of a point, the subscripts denoting the ellipsoid to which the axes belong and on the surface of which the point lies.

Hence

$$X_{23} = \frac{b_2 c_2}{b_3 c_3} X_{32} = 2 \reflectbox{∂} \rho a_3 b_2 c_2 Q_{a_3} x_2 = 2 \reflectbox{∂} \rho a_2 b_2 c_2 Q_{a_3} x_3;$$

for x_2 and x_3 being co-ordinates of corresponding points, we have

$$a_3 x_2 = a_2 x_3.$$

Now

$$X = X_{23} - X_{13} = 2\,\partial\rho Q_{a_3} x_3 \delta\,(a_1 b_1 c_1)$$
$$= 2\,\partial\rho\, Q_{a_3} x_3 \{a_1 b_1 \delta c_1 + a_1 c_1 \delta b_1 + b_1 c_1 \delta a_1\},$$

and, since

$$a^2 - b^2 = \text{const.}, \qquad a^2 - c^2 = \text{const.}, \qquad a\delta a = b\delta b = c\delta c,$$

so that

$$X = 2\,\partial\rho Q_{a_3} x_3 \left\{ \frac{a_1{}^2 b_1{}^2 + b_1{}^2 c_1{}^2 + a_1{}^2 c_1{}^2}{a_1 b_1 c_1} \right\} a_1 \delta a_1,$$
$$= k Q_{a_3} x_3,$$

or, dropping subscripts,

$$X = k Q_a x,$$
$$Y = k Q_b y,$$
$$Z = k Q_c z,$$

where

$$Q_b = \int_0^\infty \frac{dt}{(b^2 + t)\sqrt{(a^2 + t)(b^2 + t)(c^2 + t)}}$$

and

$$Q_c = \int_0^\infty \frac{dt}{(c^2 + t)\sqrt{(a^2 + t)(b^2 + t)(c^2 + t)}}.$$

Legendre has shown that

$$Q_a + Q_b + Q_c = \frac{2}{abc},$$

and the same result may also be obtained as follows. By the theory of potential,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

for all points external to the attracting mass. Hence

$$Q_a + Q_b + Q_c + x\frac{dQ_a}{dx} + y\frac{dQ_b}{dy} + z\frac{dQ_c}{dz} = 0.$$

Now

$$\frac{dQ_a}{dx} = \left\{ \frac{dQ_a}{d(a^2)} + \frac{dQ_a}{d(b^2)} + \frac{dQ_a}{(dc^2)} \right\} \frac{d(a^2)}{dx} = \frac{d}{d(a^2)}(Q_a + Q_b + Q_c)\frac{(da^2)}{dx};$$

and

$$\frac{2\,xdx}{a^2} = \frac{x^2 d(a^2)}{a^4} + \frac{y^2 d(b^2)}{b^4} + \frac{z^2 d(c^2)}{c^4} = \frac{d(a^2)}{p^2}.$$

$$\therefore \frac{d(a^2)}{dx} = \frac{2\,xp^2}{a^2}.$$

$$\therefore Q_a + Q_b + Q_c + \frac{2\,x^2p^2}{a^2}\frac{d}{d(a^2)}(Q_a + Q_b + Q_c)$$

$$+ \frac{2\,y^2p^2}{b^2}\frac{d}{d(b^2)}(Q_a + Q_b + Q_c) + \frac{2\,z^2p^2}{c^2}\frac{d}{d(c^2)}(Q_a + Q_b + Q_c) = 0,$$

for all values of x, y, and z.

$$\begin{aligned}\therefore Q_a + Q_b + Q_c &= -\,2\,a^2\frac{d}{d(a^2)}(Q_a + Q_b + Q_c)\\ &= -\,2\,b^2\frac{d}{d(b^2)}(Q_a + Q_b + Q_c)\\ &= -\,2\,c^2\frac{d}{d(c^2)}(Q_a + Q_b + Q_c).\end{aligned}$$

Integrating,

$$Q_a + Q_b + Q_c = \frac{\text{const.}}{abc}.$$

To determine this constant we must use the surface condition

$$\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2} = 4\,\reflectbox{∂}\sigma.$$

The X force which the shell exerts on a point in its interior is

$$+2\,\reflectbox{∂}\rho x\delta(a_1b_1c_1Q_{a_1}) = +\,2\,\reflectbox{∂}\rho xa_1b_1c_1\delta Q_{a_1} + 2\,\reflectbox{∂}\rho xQ_{a_1}\delta(a_1b_1c_1),$$

so that the sum of the X components on the two sides of the shell is

$$2\,\reflectbox{∂}\rho xa_1b_1c_1\delta Q_a = 2\,\reflectbox{∂}\rho xabc\left\{\frac{dQ_a}{d(a^2)} + \frac{dQ_b}{d(b^2)} + \frac{dQ_c}{d(c^2)}\right\}\delta(a^2).$$

Resolving the sums of the components along the normal, we have

$$2\,\reflectbox{∂}\rho abc\left\{\frac{d}{d(a^2)}(Q_a + Q_b + Q_c)\frac{px^2}{a^2} + \frac{d}{d(b^2)}(Q_a + Q_b + Q_c)\frac{py^2}{b^2}\right.$$

$$\left. + \frac{d(c^2)}{d}(Q_a + Q_b + Q_c)\frac{pz^2}{c^2}\right\}\delta(a^2) = -\,4\reflectbox{∂}\sigma = -\,4\,\reflectbox{∂}\theta\rho,$$

where θ is the thickness of the shell and ρ is the density of the original ellipsoids.

Now

$$\theta = \frac{a\delta a}{p},$$

for, along the normal, we have

$$\frac{y\delta x}{b^2} = \frac{x\delta y}{a^2}, \qquad \frac{z\delta x}{c^2} = \frac{x\delta z}{a^2}.$$

Also

$$\frac{x\delta x}{a^2} + \frac{y\delta y}{b^2} + \frac{z\delta z}{c^2} = \frac{x^2\delta a}{a^3} + \frac{y^2\delta b}{b^3} + \frac{z^2\delta c}{c^3}.$$

$$\therefore\ (\delta x^2 + \delta y^2 + \delta z^2)^{\frac{1}{2}} = \frac{a\delta a}{p}.$$

$$\therefore\ -abcp^2 \left\{ \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right\} \frac{\text{const.}}{2\,abc} = -1.$$

$$\therefore\ \text{const.} = 2.$$

$$\therefore\ Q_a + Q_b + Q_c = \frac{2}{abc}.$$

. . . .

Let V denote the potential due to the shell on its exterior. Then

$$dV = k\,\{Q_a x dx + Q_b y dy + Q_c z dz\}.$$

Hence for all points on the ellipsoid whose axes are $2a$, $2b$, $2c$,

$$V - V_0 = \frac{k}{2}\,\{Q_a x^2 + Q_b y^2 + Q_c z^2 - Q_a x_0^2 - Q_b y_0^2 - Q_c z_0^2\},$$

where V and V_0 are the potentials at xyz and $x_0 y_0 z_0$, both points being on the ellipsoidal surface of which the axes are $2\,a$, $2\,b$, $2\,c$.

Now let

$$x_0 = \frac{a}{\sqrt{3}}, \qquad y_0 = \frac{b}{\sqrt{3}}, \qquad z_0 = \frac{c}{\sqrt{3}},$$

so that

$$x_0 dx_0 = \frac{ada}{3}, \qquad y_0 dy_0 = \frac{bdb}{3}, \qquad z_0 dz_0 = \frac{cdc}{3}.$$

Then

$$V_0 = \frac{k}{3}\int_\infty^a \{Q_a a da + Q_b b db + Q_c c dc\} = \frac{k}{3}\,Q,$$

where

$$Q = -2\int_0^\infty \frac{dt}{\sqrt{(a^2+t)\,(b^2+t)\,(c^2+t)}};$$

$$\therefore\ V = \frac{k}{2}\left\{ Q_a x^2 + Q_b y^2 + Q_c z^2 - \frac{a^2 Q_a + b^2 Q_b + c^2 Q_c}{3} + \frac{1}{3}\,Q \right\}.$$

Also

$$Q = 2\int_{\infty}^{a}\left\{ Q_a a\,da + Q_b b\,db + Q_c c\,dc \right\} = 2\int_{\infty}^{a}\frac{2\,a\,da}{abc} = 4\int_{\infty}^{a}\frac{da}{bc},$$

and consequently Q is a homogeneous function of a, b, c of degree -1,

$$\therefore\ a^2Q_a + b^2Q_b + c^2Q_c = -\tfrac{1}{2}Q.$$

$$\therefore\ V = \frac{k}{2}\left\{ x^2Q_a + y^2Q_b + z^2Q_c + \tfrac{1}{2}Q \right\}$$

$$= \frac{k}{2}\left\{ (x^2 - a^2)\,Q_a + (y^2 - b^2)\,Q_b + (z^2 - c^2)\,Q_c \right\}.$$

. . . .

Within the shell

$$X = +\,2\,\partial\rho x Q_{a_1}\delta(a_1b_1c_1) + 2\,\partial\rho x a_1b_1c_1\delta Q_{a_1}$$

$$= \left[+\,2\,\partial\rho Q_{a_1}\left\{\frac{a_1{}^2b_1{}^2 + b_1{}^2c_1{}^2 + c_1{}^2a_1{}^2}{a_1b_1c_1}\right\}x - \frac{4\partial\rho a_1b_1c_1}{a_1{}^3b_1c_1}\,x\right] a_1\delta a_1.$$

$$X = +\,kQ_{a_1}x - 4\,\partial\rho\,\frac{xa_1\delta a_1}{a_1{}^2},$$

$$Y = +\,kQ_{b_1}y - 4\,\partial\rho\,\frac{ya_1\delta a_1}{b_1{}^2},$$

$$Z = +\,kQ_{c_1}z - 4\,\partial\rho\,\frac{za_1\delta a_1}{c_1{}^2}$$

$$V = \frac{k}{2}\,(x^2Q_{a_1} + y^2Q_{b_1} + z^2Q_{c_1}) - 2\,\partial\rho\left(\frac{x^2}{a_1{}^2} + \frac{y^2}{b_1{}^2} + \frac{z^2}{c_1{}^2}\right)a_1\delta a_1 + \text{const.}$$

At the surface the const.

$$= 2\,\partial\rho a_1\delta a_1 + \frac{k}{4}\,Q\,;$$

$$\therefore\ V = \left\{\frac{k}{2}\,Q_{a_1} - \frac{2\,\partial\rho a_1\delta a_1}{a_1{}^2}\right\}x^2 + \left\{\frac{k}{2}\,Q_{b_1} - \frac{2\,\partial\rho a_1\delta a_1}{b_1{}^2}\right\}y^2$$

$$+ \left\{\frac{k}{2}\,Q_{c_1} - \frac{2\,\partial\rho a_1\delta a_1}{c_1{}^2}\right\}z^2 + 2\,\partial\rho a_1\delta a_1 + \frac{k}{4}\,Q.$$

The equipotential surfaces within the shell are quadrics, and it is easily seen that they are hyperboloids. For if a, b, c denote their semi-axes, we have

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{k}{2}\left\{Q_{a_1}+Q_{b_1}+Q_{c_1}\right\}-2\,\partial\rho a_1\delta a_1\left\{\frac{1}{a_1{}^2}+\frac{1}{b_1{}^2}+\frac{1}{c_1{}^2}\right\}$$

$$=2\,\partial\rho\left[\left\{\frac{a_1{}^2b_1{}^2+b_1{}^2c_1{}^2+c_1{}^2a_1{}^2}{a_1{}^2b_1{}^2c_1{}^2}\right\}-\left\{\frac{a_1{}^2b_1{}^2+b_1{}^2c_1{}^2+c_1{}^2a_1{}^2}{a_1{}^2b_1{}^2c_1{}^2}\right\}\right]a_1\delta a_1=0.$$

So that any internal equipotential surface is an hyperboloid of one sheet of which the square of the imaginary axis is numerically the least, or an hyperboloid of two sheets of which the square of the real axis is numerically the least. The equations of the external equipotential surfaces involve elliptic functions.